Finite element approximation of free vibration of folded plates

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Abstract

In this paper a finite element approximation of the free vibration of folded plates is studied. Naghdi model, including bending, shear and membrane terms for the plate, is considered. Quadrilateral low order MITC (Mixed Interpolation Tensorial Component) elements are used for the bending and shear effect, coupled with standard quadratic elements enriched with a drilling degree of freedom for the membrane term. Convergence properties and optimal order error estimates are proved. Numerical examples, showing the good behavior of the method, are presented for one and two folded plates with different thickness and crank angles.

Key words: folded plates, drilling degree of freedom, MITC
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1 Introduction

This work deals with the finite element approximation of the free vibration of folded plates. These kind of structures are presented in many practical applications, such that roofs, sandwich plate cores, cooling towers, etc.. Their great interest, from an engineering point of view, is reflected in a large number of works where the structural behavior of folded plates has been studied by

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using a variety of approaches (see, for example, [12,15,16,17]). Nevertheless, to the best of the authors knowledge, the analysis of the convergence of the numerical methods can not be found in the literature.

In order to model this problem, we can start with Naghdi shell equations (see [5,8]) over each plate. These equations, when applied to plates, lead to a case where the transversal and the in-plane deformation appear uncoupled, following the first one the Reissner-Mindlin equations (see [8]), meanwhile the second one is modeled by the elasticity equation, and does not depend on the thickness of the plate (see [18]).

Each of these uncoupled problem is well studied in the literature. By one side, the elasticity problem is classic. By the other side, regarding the Reissner-Mindlin equations, it is now well understood that standard finite element methods applied to this problem produce very unsatisfactory results due to the so called locking phenomenon. Therefore, some special method based on reduced integration or mixed interpolation has to be used. Among these methods, the so called MITC ones, introduced by Bathe and Dvorkin in [4], or variants of them are very likely the most used in practice. For these methods, convergence and optimal error estimates independent of the thickness of the plate have been proved (see, for example, [3] for the load problem, and [10], and references therein, for the vibration problem).

When folded plates are considered, the membrane and bending terms are coupled, as can be clearly seen in Figure 3. Even more, since the rotations of the normal fibers appear as unknowns for the Reissner-Mindlin model, it is necessary to introduce a new unknown for the in-plane rotation (which is redundant for a single plate) called drilling degree of freedom (see [14]).

In this paper, to approximate the free vibrations of folded plates, we consider a method consisting of standard quadrilateral elements of order 1 to approximate the in-plane deformation enriched with drilling degree of freedom (see [14]), and low order quadrilateral MITC element (called MITC4) to approximate the bending term (see [10]).

The outline of the paper is as follows: in Section 2 we present and study our method to the case of one single plate. We prove optimal order error estimates for eigenfunctions and eigenvalues in the framework of the spectral approximation theory stated in [1]. In Section 3 we applied our method to a system made by two folded plates, showing optimal order error estimates for this case. In Section 4 we assess the performance of the method by computing the free vibrations of some Benchmark cases. Finally, some conclusions are given in Section 5.
2 Vibrations of a single plate

2.1 Naghdi equations

Let $\Omega \times -\frac{t}{2}, \frac{t}{2}$ be the region occupied by an undeformed elastic plate of thickness $t$, where $\Omega$ is a convex bi-dimensional domain.

Throughout this paper we use the standard notation in functional spaces, with $L^2(\Omega)$ denoting the space of functions whose integral of its square is bounded, and $H^1(\Omega)$ denoting the space of function in $L^2(\Omega)$ with generalized derivatives also in $L^2(\Omega)$. Let $\| \cdot \|_{1,\Omega}$ and $\| \cdot \|_{0,\Omega}$ be the standard norm defined on $H^1(\Omega)$ and $L^2(\Omega)$, respectively. Finally, $BC$ symbolically denotes the essential boundary condition to be imposed on each problem (probably, not always the same).

In order to describe the deformation of the plate, we consider the general classical Naghdi’s model, which is written in terms of the vector fields $u = (u_1, u_2, u_3)$, corresponding to the displacement of the mid-surface of the plate, and $\beta = (\beta_1, \beta_2)$, corresponding to the rotation of the fiber initially normal to the plate’s mid-surface. It is interesting to distinguish between membrane and transverse displacements, then we denote $\tilde{u}$ the membrane displacements, $\tilde{u} = (u_1, u_2)$.

We emphasize that, as a general rule of our notation, throughout this paper we use boldface variables to represent three-dimensional vectorial fields, as $u = (u_1, u_2, u_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, and vectorial variables to represent the two first components of them, $\tilde{u} = (u_1, u_2)$, $\tilde{\beta} = (\beta_1, \beta_2)$.

The vibration modes of the Naghdi model for plates are the solution of the following problems (see [5,13]):

Find $\omega > 0$ and $(u, \tilde{\beta}) \in [H^1(\Omega)]^5 \cap BC$ such that

$$A((u, \tilde{\beta}), (v, \tilde{\eta})) = \omega^2 B((u, \tilde{\beta}), (v, \tilde{\eta})) \quad \forall (v, \tilde{\eta}) \in [H^1(\Omega)]^5 \cap BC,$$

where the bilinear forms $A$ and $B$ are given by

$$A((u, \tilde{\beta}), (v, \tilde{\eta})) := \frac{t^3}{12} a(\tilde{\beta}, \tilde{\eta}) + ta(\tilde{u}, \tilde{\nu}) + k_s t \int_\Omega (\tilde{\nabla} u_3 - \tilde{\beta}) \cdot (\tilde{\nabla} v_3 - \tilde{\eta}),$$

$$B((u, \tilde{\beta}), (v, \tilde{\eta})) := t \int_\Omega u \cdot v + \frac{t^3}{12} \int_\Omega \tilde{\beta} \cdot \tilde{\eta},$$
with $k_s$ a correction factor for the shear term and $a \left( \vec{\beta}, \vec{\eta} \right)$ (respectively, $a \left( \vec{u}, \vec{v} \right)$) representing the linear-elasticity bilinear form,

$$a \left( \vec{\beta}, \vec{\eta} \right) := \int_{\Omega} \varepsilon \left( \vec{\beta} \right) : C : \varepsilon \left( \vec{\eta} \right).$$

Here $\varepsilon \left( \vec{\beta} \right)$ denotes the classical linear strain tensor, $\varepsilon \left( \vec{\beta} \right) = \frac{1}{2} \left( \vec{\nabla} \vec{\beta}^T + \vec{\nabla} \vec{\beta} \right)$, $C$ the material tensor defined by the Hooke’s law, and $k_s$ is a correction factor for the shear strain.

2.2 Uncoupled problems

It is clear that the well known Reissner-Mindlin model for the bending plate (see [4, 11]) can be seen as a special case of the above general formulation. In fact, if we consider only the plate transversal displacements (which, in this case, can be dealt separately from in-plane term), we get the Reissner-Mindlin plate theory, which ensures the incorporation of shear deformation effects, namely:

Find $\omega_T > 0$ and $(u_3, \vec{\beta}) \in V_T$ such that

$$\frac{t^3}{12} a \left( \vec{\beta}, \vec{\eta} \right) + k_s t \int_{\Omega} \left( \vec{\nabla} u_3 - \vec{\beta} \right) : \left( \vec{\nabla} v_3 - \vec{\eta} \right) = (\omega_T)^2 \left( t \int_{\Omega} u_3 v_3 + \frac{t^3}{12} \int_{\Omega} \vec{\beta} : \vec{\eta} \right) \quad \forall \left( v_3, \vec{\eta} \right) \in V_T, \quad (2)$$

where $V_T := \left[ H^1 \left( \Omega \right) \right]^3 \cap \mathbf{BC}$. Moreover, the in-plane deformation, or membrane terms, deals with the following standard linear elasticity problem:

Find $\omega_M > 0$ and $\vec{u} \in \left[ H^1 \left( \Omega \right) \right]^2 \cap \mathbf{BC}$ such that

$$ta \left( \vec{u}, \vec{v} \right) = (\omega_M)^2 t \int_{\Omega} \vec{u} : \vec{v} \quad \forall \vec{v} \in \left[ H^1 \left( \Omega \right) \right]^2 \cap \mathbf{BC}. \quad (3)$$

The problems above have been deeply studied in the bibliography (see [10], and reference therein, for the plate problem, and [2, 6] for the linear elasticity problem).
2.3 The drilling degree of freedom

To describe the membrane deformation (problem (3)) it is not necessary to deal with the in-plane rotations, since it can be described by means of the in-plane displacements, $u_1$ and $u_2$. Anyway, our goal is to describe systems consisting on two or more folded plates, and, in this case (as can be clearly seen in Figure 3) the in-plane rotations of a plate are transformed into normal rotations of another one.

Then, we modify the problem (3) by introducing a redundant in-plane rotation introduced by Hughes and Brezzi in [14] for linear elasticity problems, which is called drilling degree of freedom (see Figure 1).

Hence, if we denote $\text{rot } \vec{u} = \partial u_1 / \partial x_2 - \partial u_2 / \partial x_1$, the membrane equation reads in our case

\[ ta(\vec{u}, \vec{v}) + k_d \int_\Omega (\text{rot } \vec{u} - \beta_3) (\text{rot } \vec{v} - \eta_3) = (\omega_M)^2 t \int_\Omega \vec{u} \cdot \vec{v} \quad \forall (\vec{v}, \eta_3) \in V_M, \tag{4} \]

where $V_M := (H^1(\Omega))^2 \times L^2(\Omega) \cap BC$ and $k_d$ is a real parameter to be fixed.

It is important to emphasize that the drilling degree of freedom represents the rotational of the in-plane displacement. Then, since we assume that $\vec{u} \in H^1(\Omega)^2$, we have that $\beta_3 \in L^2(\Omega)$.

Joining the uncoupled problems (2) and (4), we write the Naghdi redundant problem for a plate:

\[ \text{Find } \omega > 0 \text{ and } (\vec{u}, \beta_3) \in V \text{ such that} \]

\[ A((\vec{u}, \beta_3), (v, \eta)) = \omega^2 B((\vec{u}, \beta_3), (v, \eta)) \quad \forall (v, \eta) \in V, \tag{5} \]

where $V := ([H^1(\Omega)]^5 \times L^2(\Omega)) \cap BC$.

Here the bilinear forms $A$ and $B$ are given by
\[
A((\mathbf{u}, \mathbf{\beta}), (\mathbf{v}, \mathbf{\eta})) := \frac{t^3}{12} a (\mathbf{\beta}, \mathbf{\eta}) + t a (\bar{u}, \bar{v}) + k_s t \int_\Omega (\nabla u_3 - \mathbf{\beta}) \cdot (\nabla v_3 - \mathbf{\eta}) + k_d \int_\Omega (\text{rot } \bar{u} - \mathbf{\beta}_3) (\text{rot } \bar{v} - \mathbf{\eta}_3),
\]

(6)

\[
B((\mathbf{u}, \mathbf{\beta}), (\mathbf{v}, \mathbf{\eta})) := t \int_\Omega \mathbf{u} \cdot \mathbf{v} + \frac{t^3}{12} \int_\Omega \mathbf{\beta} \cdot \mathbf{\eta},
\]

(7)

It is important to emphasize that, in the previous formulation, the drilling term, \( \mathbf{\beta}_3 \), does not appear in the mass \( B \), as can be clearly seen in matrix formulations (11) and (19).

2.4 Finite element approximation

In this section we present a finite element method to approximate, with optimal order, the spectral problems that we have presented previously.

Concerning the Reissner-Mindlin problem (2), we consider an element of the well known MITC family, the most used methods for bending plate problems. This family, introduced in [4], has a major advantage: they produce a locking free method, it is, the error remains bounded when the thickness of the plate decrease. In particular, we consider the MITC4 element, which is the lowest order element for quadrilateral meshes among the MITC family. It is based on discretizing the bending terms, \( u_3, \mathbf{\beta}_1, \) and \( \mathbf{\beta}_2 \), with usual isoparametric quadratic finite elements, and relaxing the shear term by introducing a reduction operator \( \mathbf{R} \) on that term.

Then, we assume that \( T_h \) is a family of decompositions of \( \Omega \) into convex quadrilaterals (see, for example, [6]). If \( K \) is an element in \( T_h \), we denote by \( Q_{i,j}(K) \) the space of polynomials of degree less than or equal to \( i \) in the first variable and to \( j \) in the second one.

We introduce the reduction operator \( \varphi \mapsto \mathbf{R}\varphi \), with \( \mathbf{R}\varphi|_K \in Q_{0,1}(K) \times Q_{1,0}(K) \), \( \forall K \in T_h \) (see [3,10] for more details). Note that this operator corresponds to an interpolation on the well known rotated Raviart-Thomas space (known as edge space).

Then, the finite element approximation of problem (2) with MITC4 elements reads:

\[
\text{Find } \omega_{Th} > 0 \text{ and } (u_{3h}, \mathbf{\beta}_h) \in \mathbf{V}_{Th} \text{ such that}
\]
\[
\begin{align*}
\frac{t^3}{12} a(\beta_h, \eta_h) + k_s t \int_\Omega (\nabla u_{3h} - \mathbf{R}\beta_h) \cdot (\nabla v_{3h} - \mathbf{R}\eta_h) \\
= (\omega_{T_h})^2 \left( t \int_\Omega u_{3h} v_{3h} + \frac{t^3}{12} \int_\Omega \beta_h \cdot \eta_h \right) \quad \forall (v_{3h}, \eta_h) \in V_{Th},
\end{align*}
\]

where
\[
V_{Th} := \left\{ (u_{3h}, \beta_h) \in [L_2(\Omega)]^3 / u_{3h}|_K \in Q_{1,1}(K), \right. \\
\left. \beta_h|_K \in (Q_{1,1}(K))^2, \forall K \in T_h \right\} \cap BC.
\]

This problem can be written in matrix form as
\[
\begin{bmatrix}
k_s t G & -k_s t S \\
-k_s t S' & \frac{t^3}{12} A
\end{bmatrix}
\begin{bmatrix}
u_{3h} \\
\bar{\beta}_h
\end{bmatrix}
= (\omega_{T_h})^2
\begin{bmatrix}
t M_{u_{3h}} & 0 \\
0 & \frac{t^3}{12} M_{\bar{\beta}_h}
\end{bmatrix}
\begin{bmatrix}
u_{3h} \\
\bar{\beta}_h
\end{bmatrix},
\]

where
- \(G\) represents the matrix coming from \(\int_\Omega \nabla u_{3h} \cdot \nabla v_{3h}\),
- \(A\) represents the stiffness matrix from the bilinear form \(a(\beta_h, \eta_h)\),
- \(S\) represents the matrix from \(\int_\Omega \nabla u_{3h} \cdot \mathbf{R}\eta_h\), \(S'\) its transpose matrix,
- and \(M_{\bar{\beta}_h}, M_{u_{3h}}\) represent, respectively, the mass matrices coming from \(\int_\Omega \bar{\beta}_h \cdot \eta_h\) and \(\int_\Omega u_{3h} v_{3h}\).

MITC plate elements have a solid mathematical basis. They are reliable, efficient and locking free. In particular, for the MITC4 element, a mathematical analysis of convergence is provided in [3], where uniform meshes of square elements are used. This assumption has been weakened in [10], where, by using macro-element techniques, optimal \(H^1\) and \(L_2\) error estimates are proved. However, the \(L_2\)-estimates are obtained by assuming that the meshes are formed by higher order perturbations of parallelograms (i.e., asymptotically parallelogram meshes). All these estimates are independent of the mesh size \(h\) and of the plate thickness \(t\). Moreover, in the same reference and under the same assumptions on the meshes, it is proved the following optimal estimation for the eigenmodes and eigenfrequencies of the Reissner-Mindlin spectral problem (see Theorem 5.1 in [10]):

**Theorem 1** The solution of Problem (2) consists in a sequence of positive eigenvalues, \(\omega_T\). Furthermore, let \(\omega_T\) be an eigenvalue of problem (2) with corresponding normalized eigenfunction \((u_3, \bar{\beta})\). Then, for \(h\) small enough, there
exists an eigenvalue of the approximation problem (8), $\omega_{Th}$, with corresponding normalized eigenfunction $(u_{3h}, \vec{\beta}_h)$, such that,

$$|\omega_T - \omega_{Th}| \leq Ch^2,$$

$$\|u_3 - u_{3h}\|_{\Omega} + \|\vec{\beta} - \vec{\beta}_h\|_{\Omega} \leq Ch^{2-i}, \quad i = 0, 1.$$

Concerning the membrane terms, the approximation of problem (3) with Lagrangian elements is classical and well known. Then, if we discretize the drilling degree of freedom by the rotational of Lagrangian elements (it is, by piecewise constant functions) the analysis remains classical, due to the redundancy of such problem. But, since we are interested in the folded plate problem, we should discretize all the rotations in the same finite element spaces, because the drilling in a plate turns normal rotation in another one. Since in (8) we discretize the rotations with isoparametric quadratic finite elements, we should use the same elements to discretize the drilling (as suggested in [14] for triangular elements).

Then we want to solve the finite element problem

Find $\omega_{Mh} > 0$ and $(\vec{u}_h, \beta_{3h}) \in V_{Mh}$ such that

$$ta(\vec{u}_h, \vec{v}_h) + k_d \int_{\Omega} (\text{rot} \, \vec{u}_h - \beta_{3h}) (\text{rot} \, \vec{v}_h - \eta_{3h})$$

$$= (\omega_{Mh})^2 t \int_{\Omega} \vec{u}_h \cdot \vec{v}_h \quad \forall (\vec{v}_h, \eta_{3h}) \in V_{Mh},$$

where

$$V_{Mh} := \left\{ (\vec{u}_h, \beta_{3h}) \in [L_2(\Omega)]^3 \mid \vec{u}_h|_K \in [Q_{1,1}(K)]^2, \right.$$

$$\beta_{3h}|_K \in Q_{1,1}(K), \quad \forall K \in T_h \left\} \cap BC.$$

The matrix formulation of this problem is

$$\begin{bmatrix}
  t S + k_d R & -k_d C \\
  -k_d C' & k_d M_{\beta_{3h}}
\end{bmatrix}
\begin{bmatrix}
  \vec{u}_h \\
  \beta_{3h}
\end{bmatrix}
= (\omega_{Mh})^2
\begin{bmatrix}
  t M_{\vec{u}_h} & 0 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  \vec{u}_h \\
  \beta_{3h}
\end{bmatrix},$$

where $S$ and $M_{\beta_{3h}}$ have been defined previously and

- $M_{\vec{u}_h}$ stands for the mass matrix of the membrane displacement, $\int_{\Omega} \vec{u}_h \cdot \vec{v}_h$, 

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• $\mathcal{R}$ represents the matrix coming from $\int_{\Omega} \text{rot} \, \vec{u}_h \text{rot} \, \vec{v}_h$.
• and $\mathcal{C}$ represents the matrix from $\int_{\Omega} \text{rot} \, \vec{u}_h \, \eta_{3h}$, and $\mathcal{C}'$ its transpose.

Note that this approach leads to approximate the skew-symmetric part of the strain tensor. Hence, by adapting the argument for triangles in [14] (see Remark in pp. 118 of that reference), the finite element scheme obtained is convergent and stable with respect to the parameter $k_d$, for any $k_d > 0$.

To prove a double order of convergence of the eigenvalues in this spectral problem, as we state in Theorem 2, it is necessary to extend the result in [14] to quadrilateral meshes and to prove a double order of convergence for the drilling degree of freedom. These requirements are considered in the following lemma:

**Lemma 1** Let $(\vec{f}, \theta_3) \in [L_2(\Omega)]^3$ be a given data. Let $(\vec{u}, \beta_3)$ and $(\vec{u}_h, \beta_{3h})$ be the continuous and discrete solutions coming from the source problems associated to (4) and (10), respectively. The following inequality holds:

\[
\|\vec{u} - \vec{u}_h\|_{1, \Omega} + \|\beta_3 - \beta_{3h}\|_{0, \Omega} \leq Ch, \\
\|\vec{u} - \vec{u}_h\|_{0, \Omega} + \|\beta_3 - \beta_{3h}\|_{0, \Omega} \leq Ch^2. 
\]

**Proof:**

To obtain (12) is quite straightforward by repeating the proof of Theorem 3.1 in [14], using standard arguments to finite element schemes on quadrilateral meshes (see [6]).

To prove (13), we denote by $V'_M$ the dual space of $V_M$, and by $A^k_M$ the bounded and elliptic bilinear form in (4) (see [14]),

\[
A^k_M ((\vec{u}, \beta_3), (\vec{v}, \eta_3)) := ta (\vec{u}, \vec{v}) + k_d \int_{\Omega} (\text{rot} \, \vec{u} - \beta_3) (\text{rot} \, \vec{v} - \eta_3).
\]

Since $V_{Mh} \subset V_M$, we can obtain the error equation

\[
A^k_M ((\vec{u} - \vec{u}_h, \beta_3 - \beta_{3h}), (\vec{v}_h, \eta_{3h})) = 0 \quad \forall (\vec{v}_h, \eta_{3h}) \in V_{Mh}. 
\]

For any given data $(\vec{g}, \theta_3) \in L_2(\Omega)^3 \in V_M$, let $(\vec{u}^d, \beta^d_3)$ be the unique solution of the dual source problem associated to (4), it is,

\[
A^k_M ((\vec{v}, \eta_3), (\vec{u}^d, \beta^d_3)) = \langle (\vec{v}, \eta_3), (\vec{g}, \theta_3) \rangle_{V'_M \times V_M} \quad \forall (\vec{v}, \eta_3) \in V_M. 
\]
Then, choosing as test function in (15) \((\vec{v}, \eta_3) = (\vec{u} - \vec{u}_h, \beta_3 - \beta_{3h})\), and taking into account (14), we obtain
\[
\langle (\vec{u} - \vec{u}_h, \beta_3 - \beta_{3h}), (\vec{g}, \theta_3) \rangle_{V' \times V} = A_{kM}^d \left( (\vec{u} - \vec{u}_h, \beta_3 - \beta_{3h}), (\vec{g}^d - \vec{v}_h, \beta_{3h}^d - \gamma_{3h}) \right).
\]
(16)

On the other hand, if we assume regularity for the solution \((\vec{u}^d, \beta_{3d})\), since
\[
\| (\vec{u} - \vec{u}_h, \beta_3 - \beta_{3h}) \|_{V'} = \sup_{(\vec{g}, \theta_3) \in V} \frac{\langle (\vec{u} - \vec{u}_h, \beta_3 - \beta_{3h}), (\vec{g}, \theta_3) \rangle_{V' \times V}}{\langle (\vec{g}, \theta_3) \rangle_{V}},
\]
and \(\| \vec{u} - \vec{u}_h \|_{0,\Omega} + \| \beta_3 - \beta_{3h} \|_{0,\Omega} \leq \| (\vec{u} - \vec{u}_h, \beta_3 - \beta_{3h}) \|_{V'},\) (13) follows from (16), the continuity of \(A_{kM}^d\), (12), and standard approximation properties.

Then, putting in the context of the Theorem 7.1 in [1], we have the following optimal result for the convergence of eigenfunctions and eigenvalues for the membrane problem with the drilling degree of freedom:

**Theorem 2** The solution of Problem (4) consists in a sequence of positive eigenvalues, \(\omega_M\). Furthermore, let \(\omega_M\) be an eigenvalue of problem (4) with corresponding normalized eigenfunction \((\vec{u}, \beta_3)\). Then, for \(h\) small enough, there exists an eigenvalue of the approximation problem (10), \(\omega_{Mh}\), with corresponding normalized eigenfunction \((\vec{u}_h, \beta_{3h})\), such that,

\[
|\omega_M - \omega_{Mh}| \leq Ch^2,
\]
\[
\| \vec{u} - \vec{u}_h \|_{i,\Omega} \leq Ch^{2-i}, \ i = 0, 1,
\]
\[
\| \beta_3 - \beta_{3h} \|_{0,\Omega} \leq Ch^2.
\]

Finally, we write down the finite element spectral problem of the Naghdi formulation, with drilling degree of freedom, for one single plate:

*Find \(\omega_h > 0\) and \((\vec{u}_h, \beta_h) \in V_h\) such that*

\[
A_h \left( (\vec{u}_h, \beta_h), (\vec{v}_h, \eta_h) \right) = (\omega_h)^2 B \left( (\vec{u}_h, \beta_h), (\vec{v}_h, \eta_h) \right) \quad \forall (\vec{v}_h, \eta_h) \in V_h,
\]
(17)

where

\[
V_h := \left\{ (\vec{u}_h, \beta_h) \in \left[ L_2(\Omega) \right]^6 \mid \vec{u}_h|_K \in \left[ Q_{1,1}(K) \right]^3, \beta_h|_K \in \left[ Q_{1,1}(K) \right]^3, \forall K \in T_h \right\} \cap BC,
\]
B is the mass term defined in (7) and $A_h$ is the stiffness term modified with the reduction operator,

$$A_h((u_h, \beta_h), (v_h, \eta_h)) := \frac{t^3}{12} \int_\Omega \varepsilon(\tilde{\beta}_h) : \varepsilon(\tilde{\eta}_h) + t \int_\Omega \varepsilon(\tilde{u}_h) : \varepsilon(\tilde{v}_h)$$

$$+ k_d t \int_\Omega (\nabla u_{3h} - \bar{R} \tilde{\beta}_h) \cdot (\nabla v_{3h} - \bar{R} \tilde{\eta}_h)$$

$$+ k_d \int_\Omega (\mathbf{rot} \tilde{u}_h - \beta_{3h})(\mathbf{rot} \tilde{v}_h - \eta_{3h}).$$  \hspace{1cm} (18)

With the matrices defined previously, we can write this problem as follows:

$$\begin{bmatrix}
 t \mathcal{S} + k_d \mathcal{R} & 0 & 0 & -k_d \mathcal{C}
 0 & k_s t \mathcal{G} & -k_s t \mathcal{S} & 0
 0 & -k_s t \mathcal{S}' & \frac{t^3}{12} \mathcal{A} & 0
 -k_d \mathcal{C}' & 0 & 0 & k_d \mathcal{M}_{\beta_{3h}}
\end{bmatrix}
\begin{bmatrix}
 \tilde{u}_h \\
 u_{3h} \\
 \tilde{\beta}_h \\
 \beta_{3h}
\end{bmatrix}
= (\omega_h)^2
\begin{bmatrix}
 t \mathcal{M}_{\tilde{u}_h} & 0 & 0 & 0
 0 & t \mathcal{M}_{u_{3h}} & 0 & 0
 0 & 0 & \frac{t^3}{12} \mathcal{M}_{\tilde{\beta}_h} & 0
 0 & 0 & 0 & \mathcal{M}_{\beta_{3h}}
\end{bmatrix}
\begin{bmatrix}
 \tilde{u}_h \\
 u_{3h} \\
 \tilde{\beta}_h \\
 \beta_{3h}
\end{bmatrix}. \hspace{1cm} (19)
$$

From Theorems 1 and 2, it is direct to prove the following optimal estimate for eigenvalues and eigenvectors of the redundant Naghdi problem:

**Theorem 3** The solution of Problem (5) consists in a sequence of positive eigenvalues, $\omega$. Furthermore, let $\omega$ be an eigenvalue of problem (5) with corresponding normalized eigenfunction $(u, \beta)$. Then, for $h$ small enough, there exists an eigenvalue of the approximation problem (17), $\omega_h$, with corresponding normalized eigenfunction $(u_h, \beta_h)$, such that,

$$|\omega - \omega_h| \leq Ch^2,$$

$$\|u - u_h\|_{s,\Omega} + \|\beta - \beta_h\|_{s,\Omega} \leq Ch^{2-i}, \ i = 0, 1,$$

$$\|\beta_3 - \beta_{3h}\|_{0,\Omega} \leq Ch^2.$$
3 Folded plate approximation problem

3.1 Statement of the problem

The problem of two (or more) folded plates is more difficult to establish than the previous one, although, in practice, takes approximately the same difficulty.

For simplicity, we restrict ourselves to the simplest case of two folded plates, but the following analysis can be generalized directly to the case of a finite number of folded plates.

We associate the mid-surface of each plate with a plane domain through a change of variables, as it is usual in Naghdi shells. The deformations of the plates will be described by means of local variables. With this technique, all the computing are made in bi-dimensional domains and the main difficulty is to relate the local variables in the common boundary of each pair of plates.

Let us assume, then, that we have two plain domains, $\Omega^1$ and $\Omega^2$, and two local charts $\phi^1$ and $\phi^2$ from the plane domains to the plates (see a complete example in Section 4.2).

In the case we are concerned (it is, with the charts corresponding to plates) the geometrical coefficients are already simplified in the bilinear form $a$ of the Naghdi equation (5), which remain unchanged. Then, the charts are only useful to characterize the local variables, $u^i$ and $\beta^i$, on each plate (as can be seen in the example of Section 4.2) and to relate them on the common boundary, that we denote by $\Gamma_1$ (see Figure 3).

The variational formulation of the spectral problem involving two folded plates is, then, the addition of the uncoupled spectral problem for each plate, in a variational space where the displacements and angles are rely on the common interface, it is,

Find $\omega_F \in \mathbb{R}$ and a non-zero field $\left(u^1, \beta^1, u^2, \beta^2\right) \in V^F$ such that

$$A^i \left(\left(u^i, \beta^i\right), \left(v^i, \eta^i\right)\right) = \omega_F^2 B^i \left(\left(u^i, \beta^i\right), \left(v^i, \eta^i\right)\right) \quad \forall \left(v^1, \eta^1, v^2, \eta^2\right) \in V^F,$$  \hspace{1cm} (20)

where $A^i$ and $B^i$ are the bilinear forms defined by, respectively, (6) and (7), applying over each domain $\Omega^i$, and
\[ V^F := \left\{ (u^1, \beta^1, u^2, \beta^2) \in \left[ H^1(\Omega^1) \right]^5 \times L_2(\Omega^1) \times \left[ H^1(\Omega^2) \right]^5 \times L_2(\Omega^2) : \right. \]
\[ \beta^1|_{\Gamma_i} = D^\beta \beta^2|_{\Gamma_i}, \quad u^1|_{\Gamma_i} = D^u u^2|_{\Gamma_i} \} \cap CF, \]

with \( D^\beta \) and \( D^u \) representing, respectively, the coupling conditions between angles and displacements of both plates.

Note that matrices \( D^\beta \) and \( D^u \) depend on the parametrization of the plates and on the dihedral angle. In Section 4.2 we show some example of them.

We remark that the coupling conditions involving the drilling degree of freedom (for example, \( \beta^1|_{\Gamma_1} = \beta^2|_{\Gamma_1} \)) must be understood in a distributional sense (it is, in \( H^{-1}(\Gamma_1) \)), since we are assuming that the drilling belongs to \( L_2 \). This has no practical effects, since in the finite element problem we discretize the drilling by using piecewise linear functions, and, then, this equality takes a classical sense.

### 3.2 Finite element approximation

In this subsection we discretize the problem (20) by using the finite element spaces introduced in the Section 2.4 for each plate.

Then, the discrete problem for two folded plates can be written as

Find \( \omega_{Fh} > 0 \) and a non-zero field \( (u^1_h, \beta^1_h, u^2_h, \beta^2_h) \in V^F_h \) such that

\[ A^i_h((u_h, \beta_h), (v_h, \eta_h)) = (\omega_{Fh})^2 B^i((u_h, \beta_h), (v_h, \eta_h)) \]
\[ \forall (v^1_h, \eta^1_h, v^2_h, \eta^2_h) \in V^F_h, \quad (21) \]

where \( A^i_h \) and \( B^i \) stand for the bilinear forms in Section 2.4, applied in the domain \( \Omega^i \), for \( i = 1, 2 \), and

\[ V^F_h = \left\{ (u^i_h, \beta^i_h, u^j_h, \beta^j_h) \in \left[ L_2(\Omega^i) \right]^6 \times \left[ L_2(\Omega^j) \right]^6 / u^i_h|_K \in [Q_{1,1}(K)]^3, \right. \]
\[ \beta^i_h|_K \in [Q_{1,1}(K)]^3, \quad \forall K \in T_h, \quad K \subset \Omega^i, \quad i = 1, 2, \]
\[ \text{and } \beta^1|_{\Gamma_1} = D^\beta \beta^2|_{\Gamma_1}, \quad u^1|_{\Gamma_1} = D^u u^2|_{\Gamma_1} \} \cap BC. \]

The practical implementation of this problem is straightforward from matrix formulation (19). The coupling condition between both plates can be taken
into account by performing a static condensation, identifying the corresponding degrees of freedom on the common boundary, $\Gamma_1$, according to the coupling equations $\beta^1|_{\Gamma_1} = D\beta^2|_{\Gamma_1}$ and $u^1|_{\Gamma_1} = D^u u^2|_{\Gamma_1}$.

If we assume enough regularity for our problem, by Theorem 3, it is direct to prove the following

**Theorem 4** The solution of Problem (20) consist in a sequence of positive eigenvalues, $\omega_F$. Furthermore, let $\omega_F$ be an eigenvalue of problem (20) with corresponding normalized eigenfunction $(u^1, \beta^1, u^2, \beta^2)$. Then, for $h$ small enough, there exists an eigenvalue of the approximation problem (21), $\omega_{Fh}$, with corresponding normalized eigenfunction $(u^1_h, \beta^1_h, u^2_h, \beta^2_h)$, such that,

$$
|\omega_F - \omega_{Fh}| \leq Ch^2,
$$

$$
\|u^1 - u^1_h\|_{i,\Omega} + \|\beta^1 - \beta^1_h\|_{i,\Omega} 
+ \|u^2 - u^2_h\|_{i,\Omega} + \|\beta^2 - \beta^2_h\|_{i,\Omega} \leq Ch^2 - i, \quad i = 0, 1,
$$

$$
\|\beta^3_1 - \beta^3_{h1}\|_{0,\Omega^1} + \|\beta^3_2 - \beta^3_{h2}\|_{0,\Omega^2} \leq Ch^2.
$$

4 Numerical results

In this section we present some numerical examples showing the good behavior and the numerical performance of the method that we have presented.

4.1 Numerical results for a single plate

In the first experiment we are going to present, we will check that the numerical results for one single plate do not deteriorate with the inclusion of the drilling degree of freedom.

We consider a clamped square plate, with 1m length side and 0.01m of thickness, and we take as its physical constants

- Young, $E = 200 \times 10^9$,
- Poisson, $\nu = 0.3$,
- density, $\rho = 8000$.

We have used successive refinements of a uniform mesh as that in Figure 2, the refinement parameter $N$ being the number of element edges on each side of the square.
In Table 1 we compare the vibration frequencies, in rad/sec, for the membrane problem without drilling degree of freedom (numerical solutions of problem (3)), against the solutions of the membrane problem with drilling degree of freedom (problem (10)). We also include the value of the vibration frequencies obtained by extrapolating the computed values as well as the estimated order of convergence. Such extrapolation has been obtained by means of a least square fitting. We can see that the performance of the results remains unchanged with or without the drilling, and does not depend on the choice of the positive value $k_d$, as announced in [14].

Table 1
Membrane eigenmodes for the clamped plate.

<table>
<thead>
<tr>
<th>Modo</th>
<th>Drilling/no</th>
<th>$k_d$</th>
<th>N=12</th>
<th>N=16</th>
<th>N=20</th>
<th>Order</th>
<th>Extrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No</td>
<td>—</td>
<td>18719.899</td>
<td>18682.668</td>
<td>18665.324</td>
<td>1.98</td>
<td>18634.138</td>
</tr>
<tr>
<td>1</td>
<td>Drilling $\frac{E}{2(1+\nu)}$</td>
<td>18731.598</td>
<td>18689.525</td>
<td>18669.827</td>
<td>1.96</td>
<td>18633.954</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Drilling $\frac{E}{4(1+\nu)}$</td>
<td>18742.945</td>
<td>18696.208</td>
<td>18674.230</td>
<td>1.94</td>
<td>18633.667</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Drilling $\frac{E}{1+\nu}$</td>
<td>18764.883</td>
<td>18709.195</td>
<td>18682.819</td>
<td>1.91</td>
<td>18633.169</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>No</td>
<td>—</td>
<td>22489.324</td>
<td>22361.329</td>
<td>22302.251</td>
<td>2.01</td>
<td>22197.853</td>
</tr>
<tr>
<td>2</td>
<td>Drilling $\frac{E}{4(1+\nu)}$</td>
<td>22591.245</td>
<td>22418.073</td>
<td>22338.415</td>
<td>2.02</td>
<td>22198.414</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Drilling $\frac{E}{2(1+\nu)}$</td>
<td>22692.522</td>
<td>22474.573</td>
<td>22374.461</td>
<td>2.03</td>
<td>22199.775</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Drilling $\frac{E}{1+\nu}$</td>
<td>22893.363</td>
<td>22586.943</td>
<td>22446.253</td>
<td>2.03</td>
<td>22200.667</td>
<td></td>
</tr>
</tbody>
</table>

As a second numerical experiment, we solve the complete Naghdi problem for a plate without drilling (numerical solution of problem (1)) and with drilling (problem (17)). We have taken, as usual for clamped plates, a shear correction factor $k_s = \frac{5}{6}$. The results are shown in Table 2 where it can be seen that the numbers are almost the same.

4.2 Folded plates

In this section we report numerical results corresponding to the solution of problems including two folded plates.
Table 2
Bending eigenmodes for the clamped plate.

<table>
<thead>
<tr>
<th>Modo</th>
<th>Drilling</th>
<th>( k_d )</th>
<th>N=12</th>
<th>N=16</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No</td>
<td>—</td>
<td>553.801</td>
<td>549.394</td>
</tr>
<tr>
<td>1</td>
<td>Drilling</td>
<td>( \frac{E}{2(1+\nu)} )</td>
<td>553.801</td>
<td>549.394</td>
</tr>
<tr>
<td>2</td>
<td>No</td>
<td>—</td>
<td>1162.732</td>
<td>1138.163</td>
</tr>
<tr>
<td>2</td>
<td>Drilling</td>
<td>( \frac{E}{2(1+\nu)} )</td>
<td>1162.732</td>
<td>1138.163</td>
</tr>
</tbody>
</table>

We take in all the cases the values for the physical constants used in Section 4.1, with \( k_s = \frac{5}{6} \) and \( k_d = \frac{E}{2(1+\nu)} \).

As a first numerical experiment, we test our method by reproducing the results in [12]. In this paper, a high precision composite plate-bending element is used, and the results are compared against other methods. Different crank angles and thickness ratios are considered.

Then, we consider a system made by two folded plates making a \( \frac{\pi}{2} + \alpha \) angle (see Figure 3), with \( \alpha = 0, \frac{\pi}{6}, \frac{\pi}{3} \); i.e. crank angles of 90°, 120°, and 150°, respectively. In this case, the chart for the first plate is:

\[
\phi^1 : \Omega^1 = [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^3 \\
(x, y) \quad \rightarrow \quad (x, y \cos \alpha, 1 + y \sin \alpha).
\]

Then, the local basis is:

- \( a_1^1 := \frac{\partial \phi^1}{\partial x} (x, y) = (1, 0, 0), \)
- \( a_2^1 := \frac{\partial \phi^1}{\partial y} (x, y) = (0, \cos \alpha, \sin \alpha), \)
- \( a_3^1 := a_1^1 \times a_2^1 = (0, -\sin \alpha, \cos \alpha). \)

The angles are defined as rotations between the normal fibers, in such a way that \( \theta_1^1 \) is the rotation from \( a_3^1 \) to \( a_1^1 \), \( \theta_2^1 \) the rotation from \( a_3^1 \) to \( a_2^1 \), and \( \theta_3^1 \) the rotation from \( a_1^1 \) to \( a_2^1 \).

The chart for the second plate is:

\[
\phi^2 : \Omega^2 = [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^3 \\
(x, y) \quad \rightarrow \quad (x, 0, y).
\]

In this case, the local basis is:

- \( a_1^2 := \frac{\partial \phi^2}{\partial x} (x, y) = (1, 0, 0), \)
\[ a_2^2 := \frac{\partial \phi^2}{\partial y}(x, y) = (0, 0, 1), \]
\[ a_3^2 := a_1^2 \times a_2^2 = (0, -1, 0). \]

The coupling conditions between these folded plates on \( \Gamma_1 \) are

\[
\begin{bmatrix}
  u_1^1 \\
  u_2^1 \\
  u_3^1
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & \sin \alpha - \cos \alpha \\
  0 & \cos \alpha & \sin \alpha
\end{bmatrix}
\begin{bmatrix}
  u_1^2 \\
  u_2^2 \\
  u_3^2
\end{bmatrix},
\begin{bmatrix}
  \beta_1^1 \\
  \beta_2^1 \\
  \beta_3^1
\end{bmatrix} =
\begin{bmatrix}
  \sin \alpha & 0 & -\cos \alpha \\
  0 & 1 & 0 \\
  \cos \alpha & 0 & \sin \alpha
\end{bmatrix}
\begin{bmatrix}
  \beta_1^2 \\
  \beta_2^2 \\
  \beta_3^2
\end{bmatrix}.
\]

Since \( \Omega^1 = \Omega^2 = [0, 1] \times [0, 1] \), the meshes that we consider for each plate are those in Section 4.1, labeled in the same way.

In Table 3 we show a comparison between the results with our method against the results in [12] for a cantilever thin folded plate. To show the performance of the methods we consider a coarser meshes correspond to \( N = 8 \). We assume that the system is perfectly clamped on the edge \((0, 0, y)\) and \((0, y \cos \alpha, 1 + y \sin \alpha)\), for \( y \in [0, 1] \) (see Figure 3). In this table, we present the computed frequencies in the following non-dimensional form:

\[
\omega_{Fh} \sqrt{\frac{1 - \nu^2}{\rho}}.
\]

In Figure 4 we show the deformed first and second mode for the cantilever folded plate with a crank angle of 150°.

As a second numerical experiment, we consider a folded plate, with a crank angle of 150°, having two inclined edges clamped and the other two straight edges free; i.e. clamped on the edge \((0, 0, y),(1, 0, y), (0, y \cos \alpha, 1 + y \sin \alpha),\) and \((1, y \cos \alpha, 1 + y \sin \alpha)\), for \( y \in [0, 1] \). In Table 5 and 4, respectively, we
Table 3
Comparison for a cantilever folded plate.

<table>
<thead>
<tr>
<th>Crank angle(deg)</th>
<th>method</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
<th>Mode 4</th>
<th>Mode 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>actual</td>
<td>0.0492</td>
<td>0.0978</td>
<td>0.1835</td>
<td>0.2147</td>
<td>0.3516</td>
</tr>
<tr>
<td></td>
<td>LS9RI[12]</td>
<td>0.0488</td>
<td>0.0956</td>
<td>0.1784</td>
<td>0.2071</td>
<td>0.3407</td>
</tr>
<tr>
<td>120</td>
<td>actual</td>
<td>0.0492</td>
<td>0.0950</td>
<td>0.1837</td>
<td>0.2132</td>
<td>0.3038</td>
</tr>
<tr>
<td></td>
<td>LS9RI[12]</td>
<td>0.0487</td>
<td>0.0930</td>
<td>0.1785</td>
<td>0.2056</td>
<td>0.2847</td>
</tr>
<tr>
<td>150</td>
<td>actual</td>
<td>0.0493</td>
<td>0.0826</td>
<td>0.1838</td>
<td>0.2008</td>
<td>0.2256</td>
</tr>
<tr>
<td></td>
<td>LS9RI[12]</td>
<td>0.0487</td>
<td>0.0798</td>
<td>0.1785</td>
<td>0.1868</td>
<td>0.2172</td>
</tr>
</tbody>
</table>

Fig. 4. Deformation corresponding to the first and second modes for a cantilever folded plate.

show the non-dimensional frequencies (according to (2)) for plates with thickness $0.1m$ and $0.01m$. As proved in Theorem 4, the order of convergence is, approximately, 2. In Figure 5 we show the deformed folded plates for some of these vibration modes.

Table 4
Incline edges clamped. Thickness=0.1, angle=150.

<table>
<thead>
<tr>
<th>Mode</th>
<th>N=20</th>
<th>N=28</th>
<th>N=36</th>
<th>Order</th>
<th>Extrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6711</td>
<td>0.6682</td>
<td>0.6668</td>
<td>1.81</td>
<td>0.6644</td>
</tr>
<tr>
<td>2</td>
<td>0.6970</td>
<td>0.6944</td>
<td>0.6931</td>
<td>1.62</td>
<td>0.6904</td>
</tr>
<tr>
<td>3</td>
<td>1.1735</td>
<td>1.1686</td>
<td>1.1663</td>
<td>1.72</td>
<td>1.1619</td>
</tr>
<tr>
<td>4</td>
<td>1.6153</td>
<td>1.6013</td>
<td>1.5953</td>
<td>2.08</td>
<td>1.5862</td>
</tr>
<tr>
<td>5</td>
<td>1.6719</td>
<td>1.6584</td>
<td>1.6525</td>
<td>2.02</td>
<td>1.6432</td>
</tr>
</tbody>
</table>
Table 5
Incline edges clamped. Thickness=0.01, angle=150.

<table>
<thead>
<tr>
<th>Mode</th>
<th>N=20</th>
<th>N=28</th>
<th>N=36</th>
<th>Order</th>
<th>Extrapolated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0766</td>
<td>0.0764</td>
<td>0.0763</td>
<td>1.98</td>
<td>0.0761</td>
</tr>
<tr>
<td>2</td>
<td>0.0897</td>
<td>0.0894</td>
<td>0.0893</td>
<td>1.88</td>
<td>0.089157</td>
</tr>
<tr>
<td>3</td>
<td>0.1972</td>
<td>0.1953</td>
<td>0.1946</td>
<td>2.16</td>
<td>0.19346</td>
</tr>
<tr>
<td>4</td>
<td>0.2052</td>
<td>0.2034</td>
<td>0.2026</td>
<td>2.14</td>
<td>0.20151</td>
</tr>
<tr>
<td>5</td>
<td>0.2328</td>
<td>0.2313</td>
<td>0.2307</td>
<td>2.05</td>
<td>0.22974</td>
</tr>
</tbody>
</table>

Fig. 5. Incline edges clamped. Deformation for the first and second modes.

Finally, in order to assess the quality of the method for very thin plates, we consider a folded plate with a crank angle of 120°, clamped on the whole of its boundary, with different thickness-to-span ratio. Tables 6, 7 and 8 show the three lowest computed vibration frequencies. As in the previous case, extrapolated more accurate values are included. It can be seen that the numerical results do not deteriorate when the thickness decrease, what shows that the method is free of locking. Figure 6 shows the first and third deformed mode.

<table>
<thead>
<tr>
<th>N = 20</th>
<th>N = 28</th>
<th>N = 36</th>
<th>ORD</th>
<th>Extrapolated</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2670.702</td>
<td>2648.981</td>
<td>2639.692</td>
<td>1.95</td>
</tr>
<tr>
<td>0.01</td>
<td>282.228</td>
<td>279.869</td>
<td>278.909</td>
<td>2.18</td>
</tr>
<tr>
<td>0.001</td>
<td>28.239</td>
<td>28.004</td>
<td>27.908</td>
<td>2.18</td>
</tr>
</tbody>
</table>

Table 6
Totally clamped folded plate. First mode when decreasing the thickness.
Table 7
Totally clamped folded plate. Second mode when decreasing the thickness.

<table>
<thead>
<tr>
<th></th>
<th>N = 20</th>
<th>N = 28</th>
<th>N = 36</th>
<th>ORD</th>
<th>extrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3171.929</td>
<td>3145.245</td>
<td>3134.305</td>
<td>2.14</td>
<td>3117.847</td>
</tr>
<tr>
<td>0.01</td>
<td>381.338</td>
<td>376.292</td>
<td>374.259</td>
<td>2.21</td>
<td>371.326</td>
</tr>
<tr>
<td>0.001</td>
<td>38.207</td>
<td>37.699</td>
<td>37.495</td>
<td>2.21</td>
<td>37.200</td>
</tr>
</tbody>
</table>

Table 8
Totally clamped folded plate. Third mode when decreasing the thickness.

<table>
<thead>
<tr>
<th></th>
<th>N = 20</th>
<th>N = 28</th>
<th>N = 36</th>
<th>ORD</th>
<th>extrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3865.709</td>
<td>3145.245</td>
<td>3134.305</td>
<td>1.82</td>
<td>3790.599</td>
</tr>
<tr>
<td>0.01</td>
<td>416.3157</td>
<td>412.703</td>
<td>411.229</td>
<td>2.17</td>
<td>409.055</td>
</tr>
<tr>
<td>0.001</td>
<td>41.666</td>
<td>41.305</td>
<td>41.158</td>
<td>2.18</td>
<td>40.942</td>
</tr>
</tbody>
</table>

Fig. 6. First and Third eigenmode for a totally clamped folded plate

5 Conclusions

We have considered a finite element scheme for folded plates, coupling MITC4 finite elements for the shear and bending part with standard quadratic finite elements, enriched with drilling degrees of freedom, for the membrane.

In the case of one single plate, we have proved optimal order error estimates for eigenvalues and eigenfrequencies. We have presented numerical results showing that, in practice, the optimal order is achieved and, also, showing that the inclusion of the drilling degree of freedom does not deteriorate the results.

We have extended this result to a system made by two folded plates, proving again optimal error estimates for eigenvalues and eigenfrequencies, not depending on the thickness. We have presented numerical examples for different cases of two folded plates, with different thickness, boundary conditions and crack angles. The order of convergence, in practice, approaches the theoretical
one, and the results are completely free of locking in all cases.

We would like to put into account, as a remark, that this coupled method can be used to solve shell problems when numerical locking (shear and membrane) is presented.

References


